An Exact QED_{3+1} Effective Action

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Abstract

We compute the exact QED_{3+1} effective action for fermions in the presence of a family of static but spatially inhomogeneous magnetic field profiles. An asymptotic expansion of this exact effective action yields an all-orders derivative expansion, the first terms of which agree with independent derivative expansion computations. These results generalize analogous earlier results by Cangemi et al in QED_{2+1}.

The effective action plays a central role in quantum field theory. Here we consider the effective action in quantum electrodynamics (QED) for fermions in the presence of a background electromagnetic field. Using the proper-time technique [1], Schwinger showed that the QED effective action can be computed exactly for a constant (and for a plane wave) electromagnetic field. For general electromagnetic fields the effective action cannot be computed exactly, so one must resort to some sort of perturbative expansion. A common perturbative approach is known as the derivative expansion [2, 3, 4] in which one expands formally about the constant field case, assuming that the background is 'slowly varying'. However, even first-order derivative expansion calculations of the effective action are cumbersome, and somewhat difficult to interpret physically. A complementary approach is to seek other (i.e., inhomogeneous) background fields for which the effective action can be computed exactly, with the hope that this will lead to a better nonperturbative understanding of the derivative expansion. There are two technical impediments to such an exact computation of the effective action. First, the

background field must be such that the associated Dirac operator has a spectrum that is known exactly. Second, this spectrum will (in general) contain both discrete and continuum states, and so an efficient method is needed to trace over the entire spectrum. (Note that in the constant field case the spectrum is purely discrete so this trace is a simple sum). Cangemi et al [5] used a resolvent technique to obtain an *exact* answer for the effective action in $2+1$-dimensional QED for massive fermions in the presence of static but spatially inhomogeneous magnetic fields of the form $B(x,y) = B\,\mathrm{sech}^2(\frac{x}{\lambda})$. In this Letter, we extend this result to $3+1$-dimensional QED.

Consider the QED_{3+1} effective action

$$S = -i \ln \det(i\not{D} - m) = -\frac{i}{2} \ln \det(\not{D}^2 + m^2) \tag{1}$$

where $\not{D} = \gamma^\nu(\partial_\nu + ieA_\nu)$, and A_ν is a fixed classical gauge potential with field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We work in Minkowski space, and the Dirac gamma matrices γ^ν satisfy the anticommutation relations $\{\gamma^\nu, \gamma^\sigma\} = 2\,\mathrm{diag}(1,-1,-1,-1)$. Schwinger's proper-time formalism [1] involves representing $\ln\det$ as $\mathrm{Tr}\ln$ and using an integral representation for the logarithm:

$$S = \frac{i}{2}\int_0^\infty \frac{ds}{s} Tr \exp[-is(\not{D}^2 + m^2)] \tag{2}$$

In the constant field case one can compute exactly the proper time propagator $Tr\exp[-is(\not{D}^2 + m^2)]$, leading to an exact integral representation [1] for the effective action (see below - eq (11)). Furthermore, for non-constant but 'slowly varying' fields, the first order derivative expansion contribution has been computed in [3, 4].

To make contact with the exact QED_{2+1} result of Cangemi et al [5], we restrict our attention to static background magnetic fields that point in a fixed direction (say, the x^3 direction) in space, and whose magnitude only depends on one of the spatial coordinates (say x^1). This type of configuration can be represented by a gauge field $A_\mu = (0, 0, A_2(x^1), 0)$, where A_2 is only a function of x^1. Then, it is straightforward to show that the operator $\not{D}^2 + m^2$ diagonalizes as

$$\not{D}^2 + m^2 = [m^2 + p_3^2 - p_0^2]\mathbf{1} + \mathrm{diag}(\mathcal{D}_+, \mathcal{D}_-, \mathcal{D}_-, \mathcal{D}_+) \tag{3}$$

where

$$\mathcal{D}_\pm = p_1^2 + (p_2 - eA_2(x^1))^2 \pm eB(x^1) \tag{4}$$

and $B(x^1)$ is the magnitude of the background magnetic field. These operators $\mathcal{D}_\pm$ are precisely the ones that appear in the computation of the parity-even QED_{2+1} effective action (with static magnetic background depending on just one of the spatial coordinates) - as was done in [5]. Thus, the only difference between the QED_{3+1} case described above and the computation in [5] is the appearance of an extra trace over x^3 and p_3, the momentum corresponding to the free motion in the x^3 direction (plus an extra overall factor of 2 from the Dirac trace).

In terms of the proper-time representation (2) this is a straightforward generalization - one performs the p_3 integral, which (up to overall factors) simply changes the power of s in the proper time integral from s^{-1} to $s^{-3/2}$. This much is obvious and well-known. However, the computation of the exact QED_{2+1} effective action in [5] was done using the resolvent method rather than the proper time method, so this trick is not directly applicable. Instead, we note that an alternative way to view this generalization from $2+1$ to $3+1$ is simply to take the $2+1$ expression and replace m^2 by $m^2+p_3^2$, and then integrate over x^3 and p_3, as is clear from (3). Thus, if we take the final answer from [5] and perform this operation, we obtain the exact effective action for QED_{3+1} in the background of a family of static magnetic fields of the form

$$\vec{B} = (0, 0, B\,\mathrm{sech}^2(\frac{x^1}{\lambda})) \tag{5}$$

where B is a constant setting the scale of the magnetic field strength, and λ is a length scale describing the 'width' of the inhomogeneous profile of $|\vec{B}|$ in the x^1 direction.

From [5], the exact parity-even QED_{2+1} effective action in the family of magnetic backgrounds with profile $B(x,y) = B\,sech^2(x/\lambda)$ is:

$$S_{2+1} = -\frac{L}{4\pi\lambda^2}\int_0^\infty \frac{dt}{e^{2\pi t}-1}\;\left((eB\lambda^2 - it)\frac{(\lambda^2 m^2 + v^2)}{v}\ln\frac{\lambda m - iv}{\lambda m + iv} + c.c.\right) \tag{6}$$

where $c.c.$ denotes the complex conjugate, and $v^2 \equiv t^2 + 2i\,t\,eB\lambda^2$. Here L is the length scale of the y direction, and we suppress the overall time scale as all fields are static. An asymptotic expansion of this integral for large $B\lambda^2$ corresponds physically to expanding about the uniform B field case [recall that $B\lambda^2 \to \infty$ is the limit of uniform background], and yields the following

all-orders derivative expansion [5, 6]:

$$S_{2+1}=-\frac{Lm^3\lambda}{8\pi}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{1}{2eB\lambda^2}\right)^j\sum_{k=1}^{\infty}\frac{(2k+j-1)!\mathcal{B}_{2k+2j}}{(2k)!(2k+j-\frac{1}{2})(2k+j-\frac{3}{2})}\left(\frac{2eB}{m^2}\right)^{2k+j} \tag{7}$$

where $\mathcal{B}_k$ is the k^{th} Bernoulli number [7].

To generalize these results (6,7) to $3+1$ dimensions, we need to replace m^2 by $m^2+p_3^2$, and integrate over p_3. We first do this for the all-orders derivative expansion expression (7), and return later to the generalization of the exact integral representation (6) of the effective action.

The $j=0$ and $k=1$ term in (7) must be treated separately as it is logarithmically divergent:

$$\int_{-\frac{\Lambda}{2}}^{\frac{\Lambda}{2}}\frac{dp_3}{2\pi}\left(m^2+p_3^2\right)^{-1/2}\sim\frac{1}{2\pi}\ln\frac{\Lambda^2}{m^2}\quad,\quad\Lambda\to\infty \tag{8}$$

For the remaining terms (i.e., excluding the $j=0$ and $k=1$ term)

$$\int_{-\infty}^{\infty}\frac{dp_3}{2\pi}\left(m^2+p_3^2\right)^{3/2-2k-j}=\frac{(m^2)^{2-2k-j}}{2\sqrt{\pi}}\frac{\Gamma(2k+j-2)}{\Gamma(2k+j-3/2)} \tag{9}$$

We therefore obtain an all-orders derivative expansion of the QED_{3+1} effective action for the inhomogeneous magnetic background (5):

$$\begin{aligned}S_{3+1}\;&=\;-\frac{L^2\lambda e^2B^2}{18\pi^2}\ln\frac{\Lambda^2}{m^2}\\&-\frac{L^2\lambda m^4}{8\pi^{3/2}}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{1}{2eB\lambda^2}\right)^j\sum_{k=1}^{\infty}\frac{\Gamma(2k+j)\Gamma(2k+j-2)\mathcal{B}_{2k+2j}}{\Gamma(2k+1)\Gamma(2k+j+\frac{1}{2})}\left(\frac{2eB}{m^2}\right)^{2k+j}\end{aligned} \tag{10}$$

Here it is understood that the double sum excludes the $j=0$ and $k=1$ term.

Each power in $\frac{1}{B\lambda^2}$ in (7), and therefore also in (10), corresponds to a fixed order in the derivative expansion [5, 6]. We now compare the $j=0$ and $j=1$ terms in (10) with *independent* QED_{3+1} derivative expansion calculations [1, 3, 4]. To compute the zeroth (i.e., leading) order contribution to the derivative expansion of the effective action, we take the constant field result for the effective Lagrangian (not action!) and substitute the magnetic field

(5), and then integrate over space-time in order to obtain the contribution to the action. From [1] the constant field effective Lagrangian is

$$\mathcal{L}^{(0)} = -\frac{e^2B^2}{24\pi^2}\ln\frac{\Lambda^2}{m^2} - \frac{e^2B^2}{8\pi^2}\int_0^\infty \frac{ds}{s^2}\, e^{-m^2s/(eB)}(\coth s - \frac{1}{s} - \frac{s}{3}) \tag{11}$$

Using the expansion [7]

$$\coth(\pi t) = \frac{1}{\pi t} + \frac{2t}{\pi}\sum_{k=1}^\infty \frac{1}{k^2+t^2} \tag{12}$$

it is straightforward to develop the expansion

$$\mathcal{L}^{(0)} = -\frac{e^2B^2}{24\pi^2}\ln\frac{\Lambda^2}{m^2} - \frac{m^4}{8\pi^2}\sum_{k=2}^\infty \frac{\mathcal{B}_{2k}}{2k(2k-1)(2k-2)}\left(\frac{2eB}{m^2}\right)^{2k} \tag{13}$$

Substituting $B\,\mathrm{sech}^2(x^1/\lambda)$ for B and integrating, we obtain the zeroth order (in the derivative expansion) contribution to the effective action:

$$S^{(0)} = -\frac{L^2\lambda e^2B^2}{18\pi^2}\ln\frac{\Lambda^2}{m^2} - \frac{L^2\lambda m^4}{8\pi^{3/2}}\sum_{k=2}^\infty \frac{1}{2k}\frac{\mathcal{B}_{2k}\Gamma(2k-2)}{\Gamma(2k+\frac{1}{2})}\left(\frac{2eB}{m^2}\right)^{2k} \tag{14}$$

This result agrees exactly with the $j=0$ term from (10) including the form and magnitude of the logarithmic divergence. As shown in [1], the logarithmically divergent piece corresponds to a charge renormalization.

The first-order (in the derivative expansion) contribution to the QED_{3+1} effective Lagrangian has been computed in [3, 4]:

$$\mathcal{L}^{(1)} = -e\frac{\partial_1 B\partial_1 B}{64\pi^2 B}\int_0^\infty \frac{ds}{s}e^{-m^2s/(eB)}(s\coth s)''' \tag{15}$$

Once again, using (12) this can be expanded as

$$\mathcal{L}^{(1)} = -e^2\frac{\partial_1 B\partial_1 B}{4\pi^2 m^2}\sum_{k=1}^\infty \frac{\mathcal{B}_{2k+2}}{2k-1}\left(\frac{2eB}{m^2}\right)^{2k-2} \tag{16}$$

Substituting $B(x^1) = B\,\mathrm{sech}^2(x^1/\lambda)$ and integrating, we obtain the first order (in the derivative expansion) contribution to the effective action:

$$S^{(1)} = -\frac{L^2m^2}{8\lambda\pi^{3/2}}\sum_{k=1}^\infty \frac{\mathcal{B}_{2k+2}\Gamma(2k-1)}{\Gamma(2k+\frac{3}{2})}\left(\frac{2eB}{m^2}\right)^{2k} \tag{17}$$

This agrees exactly with the $j = 1$ term from (10).

Having understood how (7) generalizes to an all-orders derivative expansion (10) of the QED_{3+1} effective action, we conclude by presenting the exact integral representation for the QED_{3+1} effective action. This is obtained from the corresponding exact expression (6) in $2+1$ dimensions by substituting m^2 with $m^2 + p_3^2$ and tracing over p_3, as before. We find:

$$S_{3+1} = -\frac{2L^2}{3\pi^2\lambda^3}\int_0^\infty \frac{dt}{e^{2\pi t}-1}\left((eB\lambda^2 - it)\frac{(\lambda^2 m^2 + v^2)^{3/2}}{v}\arcsin(\frac{iv}{\lambda m}) + c.c.\right) \tag{18}$$

where, as before, *c.c.* denotes the complex conjugate, $v^2 \equiv t^2 + 2i\,t\,eB\lambda^2$, and we have neglected terms independent of B as they cancel against the zero-field answer, and terms quadratic in B as they may be absorbed by renormalization.

The expression (18) is the *exact* QED_{3+1} effective action for fermions in the family of inhomogeneous magnetic backgrounds (5). It is interesting to note that it is not so much more complicated than Schwinger's answer (11) for the exact effective action in the constant field case. It is a straightforward exercise to check that an asymptotic expansion of this exact result (18) for large $B\lambda^2$ yields the all-orders derivative expansion (10). We regard these results as further evidence that the formal derivative expansion should be understood as an asymptotic series expansion.

Acknowledgements: This work has been supported by the DOE grant DE-FG02-92ER40716.00, and by the University of Connecticut Research Foundation. We thank Daniel Cangemi for helpful correspondence.

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